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Pigeonhole Principle

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Often in mathematics and elsewhere, some of the simplest observations lead to very interesting results and can help solve some apparently very difficult problems. The Pigeonhole Principle is one such.

If somebody asked you to prove the following, you may at first be a little puzzled and left scratching your head:

- In India, atleast there are 27 lakh people who have the same birthday.
- If you knew that human heads in India have no more than 5 lakh hairs, then in any of our India's four metropolitan cities (population each greater than half a crore), there will be 10 non-bald people with the same number of hairs on their head.
- In any group of people, there will be at least 2 persons who have the same number of friends (or enemies if you insist).
- In any group of 6 people there are either 3 mutual friends or 3 mutual strangers.
- There exists a power of 3 that ends with 001.

By the end of this article, you should be equipped to prove all these statements – and many more! All thanks to the Pigeonhole Principle.

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What is the Pigeonhole Principle?

If you have 10 pigeons and 9 pigeonholes and you distribute these pigeons randomly, one pigeonhole will have at least 2 or more pigeons. Because in the extreme case, you will put one pigeon per hole. That will take care of 9 pigeons. The 10^{th} pigeon has then to go in one of the already occupied holes. It is even embarrassing to explain the obvious and you say how trivial.

Source photo: http://en.wikipedia.org/wiki/Pigeonhole_principle

We extend this observation to a general principle: if more than m pigeons are placed in m pigeonholes, some pigeonhole will have more than one pigeon. That is, if there are n pigeons and m pigeon holes and n>m, then one of the pigeon holes will have two or more pigeons. We can be more accurate than that – one of the pigeon holes will have n/m pigeons – and since pigeons cannot be divided, you can round off the quotient of n/m to the next highest integer. Thus if you have 100 pigeons and 30 pigeonholes, one of the pigeon holes will have at least 4 pigeons. [Please note that we are not saying *every* pigeonhole will necessarily have 4 pigeons or even one pigeon. For all you know *some* pigeonholes may have even zero pigeons.]

Birthdays in India

Now let us take the first of the above problems: In India, at least there are 27 lakh people who have the same birthday. We know India has at least 1 billion population. And there are only 366 birthdays possible (counting also February 29). The number n of our 'pigeons' here is 1 billion. The number m of our 'pigeonholes' is 366. So n/m is at least 27 lakhs or if you want to be more exact 27,32,240.437… that is at least 27 lakhs, 32 thousand and 241 persons in India have the same birthday. [Please note this is not the same as saying that on any given day, say November 14, at least 27 lakhs people in India will have their birthdays. That is a different problem involving calculating probabilities, etc. But what we are saying is that on one of the 366 days, we do not know which day, we can guarantee atleast 27 lakh people will have their birthdays.]

Now, you will be able to show:

- In a group of 53 persons, two will have the birthday the same week every year.

- In a group of 13 persons, two will have their birthdays the same month.
- In a group of 8 persons, two will have their birthday on the same day of the week.
- In any group of 27 persons, the names of two persons will start with the same English letter.
- Or in any group of more than 26x26 persons, there will be two persons at least whose first name and last name initials will be the same two English alphabets.
- A busy airport sees 1500 takeoffs and/or landings per day. Two planes must take off or land within a minute of each other. [Hint: Minutes in a 24-hour day are pigeonholes and planes pigeons.]

Number of Non-Bald People with Same Amount of Hairs

Let us take the second problem on hairs on non-bald Indians. Here our n, or number of 'pigeons' is 50 lakhs (or half a crore), and the number m of 'pigeonholes', is 5 lakhs. That is non-bald people in these four metros can have one hair on their head, 2 hairs, 3 hairs, and so on up to a maximum of 500,000. That gives us 500,000 'pigeonholes' to slot into. Again we divide n/m and we have the answer.

Now you can easily solve this problem: there are one crore trees in a forest. Each tree has no more than 400,000 leaves. At least 25 trees in the forest have exactly the same number of leaves.

Friends and Strangers and Enemies in a Group

Now we come to the third problem above. First we will simplify the problem by taking only a group of 5 - this is a recommended strategy in these kinds of problems. If you find a problem baffling, chose a particular small number and see whether you can prove the assertion. We seek to prove: In any group of 5 people, there will be at least 2 persons who have the same number of friends.

So let us name the 5 persons as A, B, C, D and E. We can have 2 cases in such a group: everybody has at least one friend; or there is somebody who has no friend. We assume friendship is a two-way process: if I am your friend, you are my friend. And I cannot be my own friend although psychologists recommend it. In the first case: everybody will have 1 friend, 2 friends, 3, friends, 4 friends. Suppose we put all the first four persons, A to D in one of these categories (that is 'pigeonholes'). The fifth person E will therefore necessarily have to be in one of the 4 categories of persons having 1 to 4 friends. That proves our assertion that in a group of 5 at least 2 persons will have the same number of friends.

In the second case the 5 persons will be slotted again into 4 categories but they will be persons with 0, 1, 2 or 3 friends (Can you say in such a group where one person has no friends, why there cannot be somebody with 4 friends?). Again the number of pigeon holes is 4 and the number of pigeons (persons A to E) is five. So therefore at least 2 persons will have the same number of friends in this group of five. The argument is the same for 5, 6 or any number of persons in a group (well to be exact any number of persons greater than 1).

Please note that this argument holds for any relationships which are symmetric: like being mutual strangers or mutual enemies, etc.

Now it will be easy to solve the following problem on similar lines: In a T-20 cricket tournament, before the semi-finals, each team plays every other team exactly once. At any moment during the tournament there will be two teams which have played, up to that moment, an identical number of games.

"In any group of 6 people there are either 3 mutual friends or 3 mutual strangers"

Now having solved the previous problem, this problem will look easier. Suppose one of the persons is Kajol (could be the Kajol). All the remaining 5 persons will either know Kajol or not (a case of 5 'pigeons' and 2 'pigeonholes'). So at least 3 of the 5 persons will be either friends of Kajol or in the category of persons who are not acquainted with Kajol.

Let us first take the category of 3 persons who are friends of Kajol. If 2 of these 3 persons know each other, the two together with Kajol form a subgroup of 3 persons who are mutual friends. If none of the 3 knows each other, then we have a subgroup of 3 persons who are mutual strangers. If 2 of the 3 do not know each other: then either both of the 2 do not know the third which gives a group of 3 mutual strangers; or one of the two knows the third, which is the same as 2 of the 3 persons knowing each other and hence together with Kajol they form a triplet of mutual friends as we have already seen.

Similarly we argue for the case of when Kajol does not know at least 3 persons. And please note that this argument and result does not depend on really knowing the names of Kajol's friends in the group.

A more visual way of solving this problem is putting six dots on a sheet of paper to represent the 6 persons. Number them 1 to 6. Connect each dot to the 5 others – that means drawing 15 straight lines (can you say how 15?). Connect with a green color if they are mutual friends or connect with red if they are not. You can do this randomly. At the end of the exercise, you will have either a red triangle or a green triangle in each and every case. A green triangle means mutual friends or a red one means mutual strangers!

Some Arithmetic Problems

Let us discuss some more examples before solving the last problem.

1) **In any group of 11 numbers, you can always choose two numbers whose difference is divisible by 10.** Divisible means divisible exactly. For solving this you have to recollect that when any number is divided by n there will be n remainders – including zero. When you divide by 10, you get one of these ten remainders 0, 1, 2, 3…9. Your pigeonholes are these 10 remainders. The 11 numbers are pigeons. It follows that 2 of the 11 numbers will have the same remainder. Hence their difference is divisible by 10. For example if the 2 numbers are 64 and 34. They can be written as: 64 = (6x10) + 4; and 34 = (3x10) + 4. Their difference is 30 = (6-3) x 10, that is their difference is divisible by 10.

   Or in general if we take 2 numbers, n_1 and n_2 which give the same remainder r on division by m,
   we have: $n_1 = q_1 m + r_1$ and $n_2 = q_2\ m = r_2$. The q's stand for quotients and the r's for remainders.
   Subtracting one from the other,
   we get $n_1 - n_2 = (q_1 - q_2) \times m$, that is, their difference is divisible by m.

From this we can prove that: from any group of (n+1) numbers, we can choose two numbers whose difference is divisible by n. If I had asked you to prove this preceding sentence without telling you about pigeonhole principle, you will most probably say you have not take Math in college or studied number theory!

2) Problem: Two powers of two differ by a multiple of 2001.

Even though you have solved the previous problem, you feel how on earth we are going to solve this problem. Well this way: take 2002 powers of 2. They can be any 2002 powers. They can be 2^2 or 2^{5642}. These 2002 powers represent 2002 numbers and on division by 2001, at least two will have the same remainder. Let these 2 numbers be 2^m and 2^n. They will as per previous problem will be divisible by 2001 – that is the same thing as saying these two powers of two differ by a multiple of 2001.

Now you can try and solve these problems:

- Some multiple of 2001 is a number all of whose digits are 0's and 1's. [Hint: Consider the sequence of 2002 numbers 1, 11, 111, · · ·.].
- Some multiple of 2001 can be written by 1s only. [Hint: Extend from the previous problem.]
- Now you can show that for any positive integer n there exists a multiple of n containing only the digits 3 and 0. [Hint: Consider n+1 numbers 3, 33, 333 …]

3) Problem: There exists a power of 3 that ends with 001.

This is the last problem in our list. Take 3^m and 3^n – two powers of 3 which give the same remainder on division by 1000. Their difference can be written as $3^m - 3^n = 3^n (3^{m-n} -1)$ and is divisible by 1000. The prime factors of 1000 are 2 and 5 and therefore they cannot divide 3^n . So 3^{m-n} -1 has to be divisible by 1000 which means (3^{m-n} -1) is a multiple of 1000; therefore 3^{m-n} , a power of 3, has digits ending in 001.

The Power of Pigeonhole Principle

Now having shown you some problems that can be solved by the Pigeonhole Principle, we recite the following problems which can be solved using the Pigeonhole Principle – it shows how a simple observation can go very far. Admittedly, the choice reflects the writer's personal idiosyncrasy. Most of them are easy – if you have internalized the Pigeonhole Principle. Only some will need some other knowledge.

1) If you pick five cards from a standard pack of 52 cards, at least two will have the same suit.
2) You have 10 pairs of socks in 2 colors – say black and brown. It is dark – there is a power cut – and you are in a hurry to pick up to find a matching pair. How many socks would you need to pick to find a matching pair? [Answer: Only 3 socks.]
3) There are 25 students in a class. They get grades A, B or C in their tests. In any of their tests, at least 9 students will get the same grade.
4) Twenty-five (25) boxes of mangoes are delivered to a mandi. Each box contains the same kind of mangoes, and there are 3 possible varieties. Prove that among these boxes there are at least 9 boxes containing the same variety of mangoes.
5) If 9 people are seated in a row of 12 chairs, then some consecutive set of 3 chairs are filled with people. [Answer: The 3 empty chairs partition the filled chairs into 4 groups of consecutive filled chairs. Since $9/4>2$, some group has at least 3 chairs filled with

people.] Or if a batsman scores 9 centuries in a row of 12 ODI matches, then he must score 3 centuries in 3 consecutive matches.

6) Among any 5 points selected inside an equilateral triangle with side of length 1, there always exists a pair at distance no more than 1/2. [Hint: Subdivide the equilateral triangle into 4 smaller same-sized equilateral triangles which are the 'pigeonholes'.]
7) Seven darts are thrown onto a circular dartboard of radius 10 units. There will always be two darts which are at most 10 units apart. [Hint: Divide the circle into six equal sectors.]
8) Sachin has 11 weeks to prepare for the next world cup. He decides to practice at the nets at least one hour every day but, in order not to tire himself, he decides not to practice more than 12 hours during any calendar week. Show that there exists a succession of consecutive days during which the master blaster will have practiced for exactly 21 hours at the nets.
9) If the fraction a/b $(b>0)$ is expressed as a decimal number, then the decimal is either terminating, or repeating with a period of length no greater than $b-1$.
10) From a list of integers, a_1, a_2… a_n there are always consecutive numbers whose sum is divisible by n.
11) Lossless data compression cannot guarantee compression of all input files!
    [See http://en.wikipedia.org/wiki/Lossless_data_compression]

Suggested references (also from which some of the problems have been taken):

1) D. Fomin, S. Genkin, I. Itenberg. *Mathematical Circles (Russian Experience)*. Universities Press, Hyderabad, India, 1998.

2) http://www.cut-the-knot.org/do_you_know/pigeon.shtml